Magnetic field penetration and electron heating in weakly nonuniform plasmas

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The simultaneous magnetic field penetration and electron heating in plasmas of nonuniform density and of azimuthal cylindrical symmetry are studied. The penetration is caused by both the convective skin effect and the pressure gradient. The shock penetration of the current layer into a cold unmagnetized plasma is shown to result in equal magnetic field energy and electron thermal energy. A direct relation is shown between the heating of an electron along its orbit and the deviation from the frozen-in law. General Hugoniot relations are presented for the shock penetration into warm magnetized plasmas. The profiles of the magnetic field and of the electron thermal energy are found in the steady skin layer that is formed in the case of no penetration.

I. INTRODUCTION

The penetration of a magnetic field into plasmas is an important process in the plasma evolution. Recently, a mechanism for magnetic field penetration, which results either from a density gradient or from a cylindrical geometry, has been explored.[1-5] This penetration is of much interest, since it is expected to occur for times (between the electron and the ion cyclotron periods) and for lengths (between the electron and the ion skin depths) that are characteristic of plasmas in certain pulsed-power devices. Indeed, the penetration of the magnetic field into the plasmas is crucial to the operation of the magnetically insulated ion diode[6] and the plasma opening switch (POS).[7,8] It has been recently suggested[2,3] that this mechanism is responsible for the fast magnetic field penetration observed in POS experiments.[9]

The mechanism of magnetic field penetration is a result of the electron dynamics only, while the ions are assumed immobile. The electrons, which carry the current, decelerate as they move from a low-density region to a high-density region in a nonuniform density plasma, or as they move from a small radius to a large radius in cylindrical geometry. Due to this deceleration the magnetic field energy that is convected into the plasma is larger than the magnetic field energy that is convected out of the plasma. A net magnetic field energy is then deposited in the plasma, and the magnetic field in the plasma grows. The rate of penetration of the magnetic field is determined by the Hall field. In a recent study[4] we calculated the rate of energy dissipation during the penetration. The dissipation was found to be large and its rate was found to depend not on the resistivity that causes the dissipation but rather on the nondissipative Hall field.

In the present paper we study the evolution of the electron thermal energy during the magnetic field penetration. This is an important subject because the electron temperature significantly affects the plasma behavior. We assume that the dissipated magnetic field energy is converted into electron thermal energy. The heating of the electrons does not affect the magnetic field evolution in the cylindrical case.

In the case of a density gradient, however, the rising electron pressure affects the evolution of the magnetic field through the pressure gradient. We solve simultaneously for the evolution of the magnetic field and of the electron energy. Perhaps the problem of most interest is the magnetic field penetration into a cold unmagnetized plasma. We find for the shock penetration, that in both the density gradient case and the cylindrical case the energy in the shock downstream is equally divided between magnetic field energy and electron thermal energy. We also study the more general problem of the shock propagation into a warm magnetized plasma. We derive the Hugoniot relations that relate the magnetic field energy and electron thermal energy in the shock upstream and downstream to the shock velocity. As is often the case with shocks, the rate of dissipation and the resulting jump in the electron thermal energy do not depend on the resistivity. The resistivity, which causes the dissipation, determines only the thickness of the shock layer.

We emphasize that the shocks described here differ from magnetohydrodynamic (MHD) shocks in that the energy here is convected by the electrons in a direction that is perpendicular to the direction of propagation of the shock. The two-dimensional solution, with the radial boundary conditions, is essential to a full understanding of the problem. Here we solve only the approximated one-dimensional problem. However, a recent two-dimensional study of the cylindrical case[4] predicts the same partition of energy, as predicted in the present one-dimensional model.

We note that in the density gradient case the pressure gradient increases the shock velocity $\frac{5}{3}$ times over the previous well-known value.[1] The previous result is still valid if the dissipated energy does not go into electron heating.

In addition to the penetration into a cold unmagnetized plasma, we look at some more special limits. An interesting case is the penetration of a weak magnetic field into a warm plasma. We show that in the density gradient case the shock velocity is determined by the electron thermal energy rather than by the magnetic field energy.

In Sec. II the governing equations for the magnetic field

and the electron thermal energy are presented. In Sec. III the slab case with a density gradient is analyzed and in Sec. IV the cylindrical geometry case is analyzed similarly. In Sec. V a direct relation between the heating of an electron along its orbit and the deviation from the frozen-in law is demonstrated. In Sec. VI we study the magnetic field penetration in the case that the electrons accelerate as they move from a high-density region to a low-density region, or as they move from a large to small radius. In this case the electrons convect more magnetic field and thermal energies out of the plasma than they convect into the plasma. During the steady state, therefore, a part of the magnetic field energy, which flows axially into the plasma, is convected radially out of the plasma and a part is dissipated. The dissipated energy becomes electron thermal energy that is convected away out of the plasma as well.

In our model we have made several simplifying assumptions. The ion motion, the electron inertia, heat conduction, and the radial boundary conditions, among others, were neglected. We also assumed that the electron pressure is isotropic. It is of much interest to examine how the relaxation of these approximations affects the results.

II. THE MODEL

We study the magnetic field evolution in plasmas, where the governing equations are Faraday's law,

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E},\tag{1}$$

Ampère's law,

$$(4\pi/c)\mathbf{j} = \nabla \times \mathbf{B},\tag{2}$$

the electron momentum equation,

$$\mathbf{E} = \eta \mathbf{j} - (\mathbf{v}_e \times B)/c - \nabla p_e/en,\tag{3}$$

and the electron heat-balance equation,

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \frac{5}{3}\epsilon \mathbf{v}_e = \mathbf{E}\cdot\mathbf{j}.\tag{4}$$

Here E and B are the electric and magnetic fields, j is the current, n is the electron (or ion) density, $\mathbf{v}_e$, p_e, and ϵ are the electron flow velocity, pressure, and energy, η is the resistivity, $-e$ is the electron charge, and c is the velocity of light in vacuum. In Eq. (2) we neglected the displacement current and in Eqs. (3) and (4) the electron inertia. We also neglected the heat conduction in Eq. (4). The electron pressure is assumed to be isotropic and to satisfy

$$p_e = \tfrac{2}{3}\epsilon.\tag{5}$$

An additional assumption is that the work that is done on the plasma increases the electron thermal energy, and we neglect the energy delivered to the ions.

Before we proceed we write the Poynting theorem, which results from Eqs. (1) and (2) as

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}\cdot\mathbf{B}}{8\pi}\right) + \frac{c}{4\pi}\nabla\cdot\mathbf{E}\times\mathbf{B} = -\mathbf{E}\cdot\mathbf{j}.\tag{6}$$

The electric field energy is smaller than the magnetic field energy as a result of the neglect of the displacement current in Ampère's law. The Joule heating converts magnetic field energy into electron thermal energy. Combining Eqs. (4) and (6) we write the energy conservation as

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}\cdot\mathbf{B}}{8\pi} + \epsilon\right) + \nabla\cdot\left(\frac{5}{3}\epsilon \mathbf{v}_e + \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right) = 0.\tag{7}$$

We now make the major assumption that the process is so fast that the ions are immobile,[1-4] and therefore

$$\mathbf{j} = -en\mathbf{v}_e.\tag{8}$$

The electron density is time independent as a result of Eqs. (2) and (8). Combining Eqs. (1)–(3) and (8), we find that the equation that governs the magnetic field evolution is

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{c^2\eta}{4\pi}\nabla^2\mathbf{B} - \nabla\times\left(\frac{\mathbf{j}\times\mathbf{B}}{en}\right) + \nabla\left(\frac{c}{en}\right)\times\nabla p_e.\tag{9}$$

The first term on the right-hand side causes the resistive diffusion, the second term results from the Hall field, and the third term represents the pressure gradient. Equations (2), (4), and (8) result in

$$\frac{\partial \epsilon}{\partial t} - \frac{5}{3}\epsilon\mathbf{j}\cdot\nabla\left(\frac{1}{en}\right) - \frac{1}{en}\mathbf{j}\cdot\nabla\epsilon = \eta j^2.\tag{10}$$

We used the fact that $\nabla\cdot\mathbf{j} = 0$, which follows (2). In the next two sections we study two cases. The first is the slab case where $\mathbf{B} = \hat{e}_z B(x,y,t)$, and z is ignorable, $\partial/\partial z = 0$. The second is the cylindrical case of azimuthal symmetry $\mathbf{B} = \hat{e}_\theta B(r,z,t)$ and $\partial/\partial\theta = 0$.

III. THE SLAB CASE

In the slab case Eqs. (9) and (10) become

$$\frac{\partial B}{\partial t} + \frac{c}{e}\hat{e}_z\cdot\nabla\left(\frac{B^2}{8\pi} + \frac{2}{3}\epsilon\right)\times\nabla\left(\frac{1}{n}\right) = \frac{c^2\eta}{4\pi}\nabla^2 B,\tag{11}$$

$$\frac{\partial \epsilon}{\partial t} + \hat{e}_z\cdot\nabla B\times\left[\frac{c}{4\pi en}\nabla\epsilon + \frac{5}{3}\frac{c\epsilon}{4\pi e}\nabla\left(\frac{1}{n}\right)\right]$$
$$= \frac{c^2\eta}{(4\pi)^2}|\nabla B|^2.\tag{12}$$

We assume now that $n = n(y)$, and look for solutions in which B and ϵ depend only weakly on y. Equations (4) and (6) for the thermal and magnetic field energies are then approximated as

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial}{\partial y}\left(\frac{5}{3}\epsilon v_{ey}\right) = \eta j_y^2\tag{13}$$

and

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) + \frac{\partial}{\partial y}\left(\frac{cv_{ey}B^2}{4\pi} - \frac{2}{3}\frac{cB}{4\pi en}\frac{\partial \epsilon}{\partial x}\right) = -\eta j_y^2.\tag{14}$$

The thermal energy and the magnetic field energy change due to the y dependence of the fluxes and due to Joule heating. This y dependence is a result of the density gradient.

We write the equations in the form

$$\frac{\partial B}{\partial t} + \frac{c\alpha}{e}\frac{\partial}{\partial x}\left(\frac{B^2}{8\pi} + \frac{2}{3}\epsilon\right) = \frac{c^2\eta}{4\pi}\frac{\partial^2 B}{\partial x^2},\tag{15}$$

$$\frac{\partial \epsilon}{\partial t} + \frac{5}{3}\frac{c\alpha}{4\pi e}\epsilon\frac{\partial B}{\partial x} = \frac{c^2\eta}{(4\pi)^2}\left(\frac{\partial B}{\partial x}\right)^2,\tag{16}$$

where $\alpha \equiv (\partial/\partial y)(1/n)$. Since the dependence on y is weak,

y serves only as a parameter. The magnetic field in Eq. (15) evolves due to the three effects mentioned before. The first is the by now well-known convective skin effect,[1-5] which results from the Hall field. If the electrons decelerate or accelerate as they move in the y direction into a higher or a lower-density plasma, the magnetic field they convect is accumulated in or evacuated from the plasma. The second effect is the evolution of the magnetic field due to gradients of density and pressure normal to each other, and the third effect is resistive diffusion. The thermal energy in Eq. (16) evolves as a result of two effects. The first is the convection of energy by electrons of a varying velocity in the y direction. The convection may increase or decrease the local energy depending upon whether the electrons decelerate or accelerate. The second effect is the Joule heating, which is a source of electron thermal energy.

In the following we examine the shock-wave penetration in the combined presence of all these effects. We assume that all the variables are functions of

$$\zeta \equiv x - \lambda t. \tag{17}$$

Equation (15) is readily integrated to

$$-\lambda B + \frac{c\alpha}{e}\left(\frac{B^2}{8\pi} + \frac{2}{3}\epsilon\right) = \frac{c^2\eta}{4\pi}\frac{dB}{d\zeta} + C_1. \tag{18}$$

By expressing ϵ as a function of B, we write Eq. (16) as

$$-4\pi\lambda\frac{d\epsilon}{dB} + \frac{5}{3}\frac{c\alpha}{e}\epsilon = \frac{c^2\eta}{4\pi}\frac{dB}{d\zeta}. \tag{19}$$

We denote the values in the shock upstream as ϵ_- and B_- and those downstream as ϵ_+ and B_+. From Eqs. (18) and (19) we find that the electron energy is

$$\epsilon = \left(\frac{5}{3}\epsilon_\pm - \frac{e\lambda}{c\alpha}B_\pm\right)\exp\left(\frac{c\alpha}{4\pi e\lambda}(B - B_\pm)\right)$$
$$+ \frac{(B^2 - B^2_\pm)}{8\pi} + \frac{e\lambda}{c\alpha}B_\pm - \frac{2}{3}\epsilon_\pm. \tag{20}$$

By the subscript $\pm$ we mean that the subscript could be either $+$ or $-$ along the equation. The equation for the spatial dependence of the magnetic field is

$$\frac{c^2\eta}{4\pi}\frac{dB}{d\zeta} = \left(\frac{10}{9}\frac{c\alpha}{e}\epsilon_\pm - \frac{2}{3}\lambda B_\pm\right)$$
$$\times\exp\left(\frac{c\alpha}{4\pi e\lambda}(B - B_\pm)\right)$$
$$+ \frac{5}{3}\frac{c\alpha}{e}\left(\frac{B^2 - B^2_\pm}{8\pi}\right)$$
$$+ \frac{5}{3}\lambda B_\pm - \lambda B - \frac{10}{9}\frac{c\alpha}{e}\epsilon_\pm. \tag{21}$$

By requiring that the derivative of B is zero on both sizes of the shock we obtain the jump relations across the shock:

$$\left[-\lambda B + \frac{c\alpha}{e}\left(\frac{B^2}{8\pi} + \frac{2}{3}\epsilon\right)\right] = 0 \tag{22}$$

and

$$\left[\left(\frac{c\alpha}{e}\epsilon - \lambda B\right)\exp\left(-\frac{c\alpha B}{4\pi e\lambda}\right)\right] = 0. \tag{23}$$

Perhaps the most interesting case is the penetration of a magnetic field into an unmagnetized cold plasma, in which $B_- = \epsilon_- = 0$. If the penetrating magnetic field is B_+, it follows from the Hugoniot relations (22) and (23) that

$$\epsilon_+ = B^2_+ /8\pi \tag{24}$$

and

$$\lambda = \tfrac{5}{3}(c\alpha B_+ /8\pi e). \tag{25}$$

The magnetic field is of the form

$$B = B_+ /\{1 - \exp[\tfrac{5}{6}(\alpha B_+ /ec\eta)\zeta]\}. \tag{26}$$

The energy downstream the shock is equally divided between the magnetic field energy and the electron thermal energy. In fact, from Eq. (20) it is easily seen that this equipartition occurs also through the shock. From Eq. (14) it follows that the flux of magnetic field energy in the positive y direction is

$$-\int_{-\infty}^{\infty} dx\,\frac{c}{4\pi}BE_x = \frac{c}{4\pi en}\cdot\frac{2}{3}\cdot\frac{5}{3}\frac{B^3_+}{8\pi}.$$

The net flux into a slab between $y = y_1$ and $y = y_2$ is

$$\frac{2}{3}\cdot\frac{5}{3}\frac{c}{4\pi e}\frac{B^3_+}{8\pi}\left(\frac{1}{n(y_1)} - \frac{1}{n(y_2)}\right). \tag{27}$$

The energy that goes into building the magnetic field in this slab is

$$\int_{y_1}^{y_2} dy\,\frac{B^2_+}{8\pi}\lambda(y) = \frac{5}{3}\frac{c}{8\pi e}\frac{B^3_+}{8\pi}\left(\frac{1}{n(y_1)} - \frac{1}{n(y_2)}\right). \tag{28}$$

Therefore, three-fourths of the magnetic field energy flux into the plasma go into building the magnetic field energy and the rest is dissipated. The Joule heating is the difference between the net energy flux (27) and the deposited energy (28). The dissipated energy is therefore

$$\int_{-\infty}^{\infty} dx\int_{y_1}^{y_2} dy\,\eta j_y^2 = \frac{5}{3}\frac{1}{4\pi e}\frac{cB^3_+}{48\pi}\left(\frac{1}{n(y_2)} - \frac{1}{n(y_1)}\right). \tag{29}$$

The radial flux of thermal energy into the slab is easily found from Eq. (13) to be

$$\int_{-\infty}^{\infty} dx\int_{y_1}^{y_2} dy\,\frac{\partial}{\partial y}\left(\frac{5}{3}\epsilon v_{ey}\right)$$
$$= \frac{5}{3}\frac{1}{8\pi}\frac{c}{4\pi e}\frac{1}{3}B^3_+\left(\frac{1}{n(y_2)} - \frac{1}{n(y_1)}\right). \tag{30}$$

The rate of growth of thermal energy is the sum of the flux (30) and the Joule heating (29), and is equal to the rate of growth of the magnetic field energy in the plasma (28).

To sum up, the energy that flows into the plasma is two-thirds magnetic field energy and one-third thermal energy. Of the magnetic field energy flux three-fourths go into building the magnetic field and one-fourth into heating. Altogether, $\frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}$ goes to the magnetic field energy and $\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{2}$ into the thermal energy. Again, we find that the energy deposited in the plasma is equally divided between magnetic field energy and electron thermal energy.

The above analysis of the energy flow could also be per-

formed for the general case. For simplicity we analyzed the energy flow only in the case of penetration into a cold unmagnetized plasma.

We note also that the shock velocity obtained here is five-thirds times larger than that obtained when the electron heating is not taken into account.[1] The result obtained in the latter case is still valid if the dissipation does not cause electron heating, for example, when the dissipation results from collisions with neutrals.

Having analyzed the penetration into a cold unmagnetized plasma, we now look at the penetration into a cold but magnetized plasma, where $(B_+ - B_-)/B_- \ll 1$. In this case we find that

$$\lambda = (c\alpha/8\pi e)(B_+ + B_-). \tag{31}$$

The change in the thermal energy is smaller than the change in the magnetic field energy, and the ratio of thermal energy change to magnetic field energy change is proportional to $(B_+ - B_-)^2/B_-^2$. The profile of the magnetic field is

$$B = B_- + \frac{(B_+ - B_-)}{\{1 + \exp[(B_+^2 - B_-^2)\alpha\zeta/4c\eta eB_-]\}}. \tag{32}$$

Finally, we discuss the penetration into a warm unmagnetized plasma where $B_- = 0$ and $B_+^2 \ll \epsilon_-$. In this case

$$\epsilon_+ = \epsilon_- + \frac{3}{2}\left(\frac{5\epsilon_-}{18\pi}\right)^{1/2} B_+ - \frac{3}{2}\frac{B_+^2}{8\pi} \tag{33}$$

and

$$\lambda = (c\alpha/e)(5\epsilon_-/18\pi)^{1/2}. \tag{34}$$

The magnetic field profile is

$$B = \frac{B_+}{\{1 - \exp[\frac{4}{5}(\alpha B_+/ec\eta)\zeta]\}}. \tag{35}$$

The change in the magnetic field energy across the shock is small relative to the change in the electron thermal energy. The ratio of these changes is proportional to $B_+/\epsilon^{1/2}$. The shock velocity is determined by the thermal energy and not by the magnetic field. The dominant effect here is the pressure gradient and not the convective skin effect.

IV. CYLINDRICAL GEOMETRY

We turn now to the case of cylindrical geometry and uniform density. Since the density is uniform the effect of the pressure gradient is absent here. The electron heating does not affect the evolution of the magnetic field. We first calculate the magnetic field. The calculated magnetic field is then used to calculate the heating. Eq. (9) for $b \equiv -rB_\theta$ becomes

$$\frac{\partial b}{\partial t} + \frac{cb}{2\pi ner^2}\frac{\partial b}{\partial z} = \frac{c^2\eta}{4\pi}\frac{\partial^2 b}{\partial z^2}. \tag{36}$$

On the right-hand side we assumed $\partial b/\partial z \gg \partial b/\partial r$. The heat balance equation for $W \equiv r^2\epsilon$ becomes

$$\frac{\partial W}{\partial t} + \left(\frac{c}{4\pi ner}\right)\hat{e}_\theta \cdot \nabla b \times \nabla W + \left(\frac{c}{2\pi ner^2}\right) W\frac{\partial b}{\partial z}$$

$$= \frac{\eta c^2}{(4\pi)^2}|\nabla b|^2. \tag{37}$$

Let us assume that b and W are mainly functions of z and the second term therefore vanishes. Equation (37) is then approximated as

$$\frac{\partial W}{\partial t} + \left(\frac{c}{2\pi ner^2}\right)W\frac{\partial b}{\partial z} = \frac{\eta c^2}{(4\pi)^2}\left(\frac{\partial b}{\partial z}\right)^2. \tag{38}$$

We look for shock solutions where the independent variable is $\zeta = z - \lambda t$. Equation (38) becomes

$$-\lambda\frac{dW}{db} + \frac{cW}{2\pi ner^2} = \frac{\eta c^2}{(4\pi)^2}\frac{db}{d\zeta}. \tag{39}$$

We first solve Eq. (36) for the magnetic field,

$$b = \frac{\{b_+ - b_-\exp[(b_+ - b_-)\zeta/necr^2\eta]\}}{\{1 - \exp[(b_+ - b_-)\zeta/necr^2\eta]\}}. \tag{40}$$

The shock velocity is

$$\lambda = c(b_+ + b_-)/4\pi ner^2. \tag{41}$$

The energy is readily found to be

$$W = \left(W_- - \frac{b_-(b_+ + b_-)}{8\pi}\right)$$

$$\times\exp\left(\frac{2(b - b_-)}{(b_+ + b_-)}\right) + \frac{b_- b_+}{8\pi} + \frac{b^2}{8\pi}. \tag{42}$$

The most interesting problem in the cylindrical case, as in the slab case, is the penetration into a cold unmagnetized plasma. The energy in the shock downstream is evenly divided between magnetic field energy and electron thermal energy similarly to the density gradient case. As we have shown previously[4] one-quarter of the magnetic field energy flux goes into heating. The radial Poynting flux is $(cb_+^2/24\pi enr^2)$. The radial flux of thermal energy is half of that, i.e., $(cb_+^3/48\pi enr^2)$. When the dissipation is taken into account, we get the equal deposition of energy into magnetic field energy and electron thermal energy. Thus, this equal partition of energy occurs both for the slab density gradient case and for the cylindrical case.

The shock solution dictates a specific relation between the radial flows of magnetic field energy and thermal energy. The question rises as to how various radial boundaries affect the radial flow from the cathode. We have recently carried out a two-dimensional analysis of a cold unmagnetized plasma in a cylindrical geometry,[4] where the heating at the cathode was shown to convert one-third of the incoming magnetic field energy into thermal energy, the same relation that is dictated by the shock solution given above. A two-dimensional analysis that takes into account the radial boundary conditions is necessary for the other cases as well.

We examine two additional cases. First we assume that $W_- = 0$ and $(b_+ - b_-)/b_- \ll 1$. In this case the thermal energy across the shock is

$$W = \frac{(b - b_-)^2}{8\pi(b_+ + b_-)}\left(b_+ - b_- - \frac{4}{3}\frac{b_-(b - b_-)}{(b_+ + b_-)}\right). \tag{43}$$

The ratio of thermal energy gain to magnetic field energy gain is approximately $\frac{1}{6}[(b_+ - b_-)/b_-]$.

As a last example we look at the shock wave that penetrates into a hot unmagnetized plasma. We assume $b_- = 0$. The energy is

$$W = W_- \exp(2b/b_+) + b^2/8\pi. \qquad (44)$$

The energy downstream is

$$W_+ = W_- \exp(2) + b_+^2/8\pi. \qquad (45)$$

The ratio of the change in the magnetic field energy to the change in the thermal energy is $b_+^2 / [8\pi W_- \exp(2)]$.

We derived the general Hugoniot relations for the shocks. The penetration into a cold unmagnetized plasma might be relevant for laboratory plasmas, as that in the POS, and for other weakly nonuniform plasmas. The occurrence of the other cases in less clear. A uniform pressure but a nonuniform density imply a nonuniform temperature. The penetration described here is of interest if it is faster than processes of heat convection, which make the temperature uniform.

V. THE FROZEN-IN LAW AND ELECTRON HEATING

The deviations from the frozen-in law and the electron heating both result from the resistivity. A detailed discussion of the deviations from the frozen-in law is given elsewhere.[10] Here we show an example where the deviation from the frozen-in law is related to the rate of heating per electron in a cylindrical geometry.

Combining Eq. (9) with the continuity equation,

$$\frac{dn}{dt} = -n\nabla\cdot\mathbf{v}_e, \qquad (46)$$

we obtain (for a cylindrical geometry)

$$\frac{d}{dt}\left(\frac{b}{nr^2}\right) = \frac{c^2\eta}{4\pi nr^2}\left[\frac{\partial^2 b}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial b}{\partial r}\right)\right]$$

$$- \frac{c}{enr}\hat{e}_\theta\cdot\nabla\left(\frac{1}{n}\right)\times\nabla p_e. \qquad (47)$$

Here $d/dt \equiv \partial/\partial t + \mathbf{v}_e\cdot\nabla$ is the convective derivative along an electron orbit, and $b \equiv -rB_\theta$. When the right-hand side vanishes, b/nr^2 is constant along an electron orbit. This is the familiar frozen-in law in cylindrical geometry.

Let us look at the uniform density case, where b and p_e are functions mainly of z. Equation (47) is approximated as

$$\frac{d}{dt}\left(\frac{b}{nr^2}\right) = \frac{c^2\eta}{4\pi nr^2}\frac{\partial^2 b}{\partial z^2}. \qquad (48)$$

We look at the shock penetration into an unmagnetized plas-

ma. The deviation from the frozen-in law is found by integrating along an electron orbit across the shock. Using $dr/dt = -j_r/en$ and $\lambda = cb_+/4\pi ner^2$ [Eq. (41)], we obtain

$$\left[\frac{b}{nr^2}\right] = \frac{8\pi}{b_+}\int_{-\infty}^{\infty}\frac{dt}{n}\eta j^2. \qquad (49)$$

Thus, the deviation from the frozen-in law is proportional to the rate of heating per electron. The rate of heating per electron is thus

$$\int_{-\infty}^{\infty}\frac{dt}{n}\eta j^2 = \frac{B_+^2}{8\pi n}. \qquad (50)$$

As previously, the electron thermal energy in the shock downstream equals the magnetic field energy.

VI. THE CASE OF NO PENETRATION

When the electrons accelerate as they move from a high-density to a low-density plasma or from a large radius to a small radius, the convective skin effect does not exist, and the magnetic field does not penetrate into the plasma. Moreover, we have even shown that the magnetic field is expelled from an initially magnetized plasma.[3,4] Here we analyze the steady skin layer. In the skin layer there is a balance between the axial energy flow into the plasma due to the large skin current and the energy that is convected radially. The density nonuniformity or the cylindrical geometry cause the electrons to convect more energy out of the plasma than into the plasma.

Let us start with the nonuniform density slab case. For the steady state Eqs. (15) and (16) become

$$\frac{c\alpha}{e}\frac{\partial}{\partial x}\left(\frac{B^2}{8\pi} + \frac{2}{3}\epsilon\right) = \frac{c^2\eta}{4\pi}\frac{\partial^2 B}{\partial x^2}, \qquad (51)$$

$$\frac{5}{3}\frac{c\alpha}{e}\epsilon = \frac{c^2\eta}{4\pi}\frac{\partial B}{\partial x}. \qquad (52)$$

The boundary conditions are

$$B(x=0) = B_0, \quad B(x=\infty) = B_\infty. \qquad (53)$$

Equations (51) and (52) combine to

$$\frac{5}{6}\frac{\alpha}{ec}\frac{\partial}{\partial x}B^2 = \eta\frac{\partial^2 B}{\partial x^2}. \qquad (54)$$

With the second boundary condition we obtain

$$\frac{dB}{dx} = \frac{5\alpha}{6ec\eta}(B^2 - B_\infty^2). \qquad (55)$$

The magnetic field is therefore

$$B = B_\infty\left(\frac{1 + [(B_0 - B_\infty)/(B_0 + B_\infty)]\exp(5B_\infty\alpha x/3ec\eta)}{1 - [(B_0 - B_\infty)/(B_0 + B_\infty)]\exp(5B_\infty\alpha x/3ec\eta)}\right); \quad x\geqslant 0. \qquad (56)$$

If the penetration is into an unmagnetized plasma, $B_\infty = 0$, the magnetic field profile becomes

$$B = \frac{B_0}{[1 - (5\alpha B_0/6ec\eta)x]}; \quad x\geqslant 0. \qquad (57)$$

With Eqs. (52) and (55) the energy at the steady state is found to be

$$\epsilon = [(B^2 - B_\infty^2)/8\pi]. \qquad (58)$$

Let us examine the energy flow into a semi-infinite plas-

ma slab located between the planes $y = y_1$ and $y = y_2$. The x component of the Poynting vector flux per unit length at the plasma boundary is

$$\int_{y_1}^{y_2} dy \frac{c}{4\pi} BE_y$$

$$= \int_{y_1}^{y_2} \frac{dy\, c^2 \eta}{32\pi^2} \frac{\partial B^2}{\partial x} \, (x=0)$$

$$= \frac{5}{96} \frac{cB_0}{\pi^2 e} (B_0^2 - B_\infty^2) \left(\frac{1}{n(y_2)} - \frac{1}{n(y_1)} \right). \quad (59)$$

Note that $n(y_2) > n(y_1)$ but the magnetic field is negative and therefore the flux is positive. In the derivation we used Eq. (55). Here there is no dependence on the resistivity. The skin layer becomes narrow if the resistivity is small and the flux remains finite. The flow into the slab in the y direction is

$$\int_0^\infty dx \frac{c}{4\pi} BE_x \Big|_{y_1}^{y_2}$$

$$= \frac{c}{4\pi e} \frac{1}{n} \Big|_{y_1}^{y_2} \int_0^\infty \frac{dx\, B}{8\pi} \frac{\partial}{\partial x} \left(B^2 + \frac{2}{3} (B^2 - B_\infty^2) \right)$$

$$= -\frac{5c(B_0^3 - B_\infty^3)}{144\pi^2 e} \left(\frac{1}{n(y_2)} - \frac{1}{n(y_1)} \right). \quad (60)$$

The energy dissipation is

$$\int_{y_1}^{y_2} dy \int_0^\infty dx \, E_y j_y$$

$$= \frac{5}{96} \frac{c}{\pi^2 e} \left(\frac{1}{3} (B_\infty^3 - B_0^3) \right.$$

$$\left. - B_\infty^2 (B_\infty - B_0) \right) \left(\frac{1}{n(y_2)} - \frac{1}{n(y_1)} \right). \quad (61)$$

The sum of the energies in Eqs. (57), (60), and (61) is of course zero. When $B_\infty = 0$, one-third of the energy that flows axially into the plasma is dissipated and two-thirds are convected away out of the plasma. When $B_0 \cong B_\infty$, most of the energy that flows into the plasma is convected away and the rate of dissipation is small.

In the flow of electron thermal energy there is a balance between the dissipated magnetic field energy and the energy that is convected out of the plasma.

We now turn to cylindrical geometry. Equations (36) and (38) for the steady state become

$$\left(\frac{c}{4\pi ner^2} \right) \frac{\partial b^2}{\partial z} = \frac{c^2 \eta}{4\pi} \frac{\partial^2 b}{\partial z^2}, \quad (62)$$

and

$$\frac{cW}{2\pi ner^2} = \frac{c^2 \eta}{(4\pi)^2} \frac{\partial b}{\partial z}, \quad (63)$$

with the boundary conditions

$$b(z=0) = b_0, \quad b(z=\infty) = b_\infty. \quad (64)$$

The magnetic field is

$$b = b_\infty \left(\frac{1 + [(b_0 - b_\infty)/(b_0 + b_\infty)] \exp(2b_\infty z/ec\eta nr^2)}{1 - [(b_0 - b_\infty)/(b_0 + b_\infty)] \exp(2b_\infty z/ec\eta nr^2)} \right); \quad z \geqslant 0. \quad (65)$$

If $b_\infty = 0$, the magnetic field profile is

$$b = b_0/(1 - 2b_0 z/ec\eta nr^2); \quad z \geqslant 0. \quad (66)$$

The electron thermal energy is

$$W = [(b^2 - b_\infty^2)/8\pi]. \quad (67)$$

We look now at the energy flow into a semi-infinite plasma slab located for $z \geqslant 0$ between $r = r_1$ and $r = r_2 > r_1$. The magnetic field energy flow into the plasma is

$$-\frac{c}{2} \int_{r_1}^{r_2} dr\, bE_r = -\frac{c}{2} \int_{r_1}^{r_2} dr\, b \left[\frac{c\eta}{4\pi r} \frac{\partial b}{\partial z} \right.$$

$$\left. - \frac{1}{en} \frac{\partial}{\partial r} \left(\frac{2}{3} \frac{W}{r^2} \right) \right]$$

$$= \frac{5cb_0(b_0^2 - b_\infty^2)}{48\pi en} \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right). \quad (68)$$

The radial flow in the plasma is

$$\frac{c}{2} \int_0^\infty dz\, bE_z = \frac{c}{2} \int_0^\infty dz\, b \left[\frac{j_r B}{enc} - \frac{2}{3en} \frac{\partial}{\partial z} \left(\frac{W}{r^2} \right) \right]. \quad (69)$$

The net radial energy flow out of the plasma is

$$-\frac{5c(b_\infty^3 - b_0^3)}{72\pi en} \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right). \quad (70)$$

The rate of dissipation is

$$\int_0^\infty dz \int_{r_1}^{r_2} dr\, 2\pi r E_r j_r$$

$$= -2\pi \cdot \frac{5}{3} \frac{c}{32\pi^2 en} \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right)$$

$$\times [\tfrac{1}{3}(b_\infty^3 - b_0^3) - b_\infty^2 (b_\infty - b_0)]. \quad (71)$$

Again, the energy pumped into the plasma (69) is partially convected radially (70) and partially dissipated (71). If $b_\infty = 0$ one-third is dissipated and two-thirds convected away.

VII. CONCLUSIONS

In this paper we have studied the simultaneous magnetic field penetration and electron heating in plasmas, where the time scale of the processes is so short that the ions are immobile. The two opposite behaviors, the fast wave propa-

gation due to the convective skin effect, and the small penetration into a narrow skin layer, have been analyzed for the two configurations: a plasma slab of a nonuniform density and a cylindrical plasma of azimuthal symmetry. General Hugoniot relations have been derived for the shock penetration into a warm magnetized plasma. The major novel results presented in this paper are that the energy in the downstream of the shock penetration into a cold unmagnetized plasma is evenly divided between magnetic field energy and electron thermal energy, the enhanced shock velocity due to the combination of the convective skin effect and the pressure gradient, the dependence of the shock velocity on the electron thermal energy in the warm plasma case, and the relation between the deviation from the frozen-in law to the heating of an electron along its orbit.

The phenomena described here could be relevant to plasmas in pulsed-power devices such as the POS, and also to other nonuniform laboratory and space plasmas.

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